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Perturbation theory for the one-dimensional Schrödinger scattering problem

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Abstract. A perturbation theory is constructed within the framework of the linear form of the variable phase approach. This allows one to correctly take into account the potential tail which decreases more rapidly than the centrifugal tail. It is shown how one can use this theory for an analysis of the scattering problem in the low-energy limit and in the limit of large angular momentum.

1. Introduction

Many versions [1–9] of perturbation theory for the one-dimensional Schrödinger scattering problem are, typically, constructed using iteration schemes [10, 11]; their constructions have some fundamental defects. The first defect is physical and mathematical incompleteness. Usually perturbation theory is realized for investigating one function only (for example, the regular wavefunction [1–5]) or one scattering characteristic (t-matrix [2–4], amplitude [5], phase-shift [6–9]). Often a construction is given only for zero angular momentum l and for a fixed total energy E. Moreover, a condition guaranteeing the uniform convergence of iteration is not investigated in physically interesting limits ($l \to \infty$ and $E \to 0$) [2, 3, 7, 8] or is not derived at all as, for instance, in [9]. However, as was demonstrated by Peierls [6] without such a condition a perturbation theory may be incorrect and, therefore, its use makes no sense. Another defect of many perturbation theories is a lack of explicit and general estimates of the convergence rates of iterations. Usually the fact of convergence is proved by the derivation of a qualitative estimate [1–5] containing the, so-called, sufficiently large but *unknown* constant. More often the convergence is only demonstrated by a particular numerical example [7–9], or the estimate of its rate is derived for a model potential [6–8] having a very simple form. Finally, in modern scattering theory [3, 4] there is no perturbation theory which is an effective method for a construction of the irregular wavefunction and which is an asymptotical method in the two limits $l \to \infty$ or $E \to 0$.

This paper aims to construct and analyse a new and complete perturbation theory which is free from the defects mentioned above.

For completeness, we briefly describe the main stages of construction for a typical problem of nuclear physics.

$$\left(\partial_x^2 - l(l+1)x^{-2} - V_c(x) - V(x) + q^2\right) u_l^{\pm}(x,q) = 0 \qquad x \in \mathbb{R}^+ \tag{1a}$$

$$u_l^{\pm} = \mathrm{O}(x^{\pm(l+1/2)+1/2}) \qquad x \to 0 \tag{1b}$$

$$u_l^{\pm}(x,q) \to \sin\left(\rho - \eta ln 2\rho - (2l+1\mp 1)\pi/4 + \delta_{cl}(q) + \delta_l(q)\right) \qquad x \to \infty \tag{1c}$$

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with the parameters $l, q \in \mathbb{R}^+$, the Coulomb potential $V_c \equiv \text{sign}\, R/x$ and the potential V obeying the sufficiently general condition

$$I_l(b,x) \equiv (2\pi/(2l+1))^{1/2} \int_b^x t|V(t)|\,dt < \infty \qquad 0 \leqslant b \leqslant x \leqslant \infty. \tag{2}$$

We use the system of units in which $\hbar = 2\mu = 1$, and we introduce the dimensionless variable $x \equiv r/|R|$ and parameter $q \equiv k|R|$ instead of the distance r and momentum k respectively. By definition, $R \equiv \hbar^2/2\mu Z_1 Z_2 e^2$ is the Bohr radius, $\rho \equiv kr = qx$ and $\eta \equiv 1/2kR = \text{sign}\, R/2q$ are the standard arguments of the Coulomb functions F_l and G_l [1, 12]; u_l^+, u_l^- and δ_l denote, respectively, the sought regular and irregular wavefunctions and the scattering phase generated by an interference between the Coulomb potential and the potential V in addition to the pure Coulomb phase δ_{cl}. Further, the index l is omitted where possible and if it is not specified, we assume that $x \in \mathbb{R}^+$, $\rho = qx$, $\eta = \text{sign}\, R/2q$ and that b denotes a certain fixed value of x.

2. Perturbation theory

2.1. Reformulation of problem (1)

To construct u_l^+ we apply the known linear form [13] (in fact it is equivalent to the method of varying constant coefficients [10]) of the variable phase approach [7, 8]. We develop this linear form by the addition of a new and simple way for constructing u_l^-, which we now describe in more detail.

Let $c^\pm$ and $s^\pm$ be amplitude functions [7] (or 'constant' coefficients [10]) obeying the Lagrange identities:

$$F(\rho,\eta)\partial_x c^\pm(x,q) + G(\rho,\eta)\partial_x s^\pm(x,q) \equiv 0. \tag{3}$$

We are looking for u^+ and then for u^- in the form

$$u^\pm(x,q) = N^\pm(q)U^\pm(x,q) + \begin{Bmatrix} 0 \\ \alpha(q)u^+(x,q) \end{Bmatrix} \tag{4a}$$

$$U^\pm(x,q) \equiv c^\pm(x,q)F(\rho,\eta) + s^\pm(x,q)G(\rho,\eta). \tag{4b}$$

Using the method [10, 13] based on the substitution of $u^\pm$ in the form of (4a) into (1a) and subsequent use of the Wronskian relation [12]

$$G(\rho,\eta)\partial_x F(\rho,\eta) - F(\rho,\eta)\partial_x G(\rho,\eta) \equiv q$$

and the identity (3), we obtain the two sets of equations

$$\partial_x \begin{Bmatrix} c^\pm(x,q) \\ s^\pm(x,q) \end{Bmatrix} = q^{-1}V(x)U^\pm(x,q) \begin{Bmatrix} +G(\rho,\eta) \\ -F(\rho,\eta) \end{Bmatrix}. \tag{5a}$$

We complete them by introducing the corresponding boundary conditions

$$\begin{Bmatrix} c^+(x,q) \\ s^+(x,q) \end{Bmatrix} \to \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + q^{-1}\int_0^x V(t)F(\rho,\eta) \begin{Bmatrix} +G(\rho,\eta) \\ -F(\rho,\eta) \end{Bmatrix} dt \tag{5b}$$

$$\begin{aligned} c^-(x,q) &\to c^-(x_0,q) + q^{-1}\int_{x_0}^x V(t)G^2(\rho,\eta)\,dt \\ s^-(x,q) &\to -q^{-1}\int_0^x V(t)F(\rho,\eta)G(\rho,\eta)\,dt \end{aligned} \tag{5c}$$

which ensure the asymptotics (1*b*) of functions (4*a*). In (5*b*) and (5*c*) $x \to 0$, $\rho = tq$ and if $V(x)G^2(\rho, \eta) \not\subset L^1_{[0,b]}$ then $x_0 = 0$ and $c^-(0, q) = 0$, otherwise x_0 is an arbitrary but fixed parameter such that $x < x_0$ and $x_0 q \ll 1$ and then c^-, s^- are constructed as follows.

Let $V(x)G^2(\rho, \eta) \not\subset L^1_{[0,b]}$. Then, due to (5*b*) and (5*c*), $| c^- | \to \infty$ as $x \to 0$, while c^+ and $s^\pm$ are always finite. We find c^+, s^+ and s^- at $x = x_0$ by solving problem (5) for c^+ and s^+ in the interval $[0, x_0]$, and by using formula (5*c*) for s^-. We then substitute these values into the Wronskian relation of problem (5)

$$W(x, q) \equiv c^+(x, q)s^-(x, q) - c^-(x, q)s^+(x, q) \equiv 1 \tag{6a}$$

stated at $x = x_0$, and resolve the equation derived with respect to $c^-(x_0, q)$. Now, we use $c^-(x_0, q)$ and $s^-(x_0, q)$ to define the functions c^- and s^- at $x \leqslant x_0$ explicitly by (5*c*) and determine them at $x \geqslant x_0$ as a solution of equations (5*a*) with the boundary conditions at $x = x_0$.

Using the described construction and the known theorems [10] we prove that under condition (2) problems (5) are uniquely solvable in the $\mathbb{C}^1_{[0,\infty]}$-class of functions and that the solutions satisfy (6*a*) and the relations

$$|c^\pm(x, q)| + |s^\pm(x, q)| > 0 \qquad |c^+(x, q)|, |s^\pm(x, q)| < \infty \tag{6b}$$

$$|c^-(x, q)| < \infty \qquad x > 0. \tag{6c}$$

Due to these facts and equations (5*b*), (5*c*) and (6), each of the functions

$$\delta(x, q) \equiv \arctan\left(s^+(x, q)/c^+(x, q)\right) \tag{7a}$$

$$N^\pm(x, q) \equiv \left(\left(c^+(x, q)\right)^2 + \left(s^+(x, q)\right)^2\right)^{\mp 1/2} \tag{7b}$$

$$\alpha(x, q) \equiv -c^+(x, q)c^-(x, q) - s^+(x, q)s^-(x, q) \tag{7c}$$

is unique and limited everywhere and, therefore, has the finite limit

$$A(q) \equiv \lim_{x \to \infty} A(x, q) \tag{8}$$

where $A = \delta, N^\pm, \alpha$. Using these properties of functions (7) the identity (6*a*) and the known asymptotics of the Coulomb functions as $x \to \infty$, we show that the wavefunctions (4*a*) will have the required asymptotics (1*c*) if we define the scattering phase $\delta(q)$ and normalization factors $N^\pm(q)$ and $\alpha(q)$ as the limits (8) of the relevant functions (7).

It should be noted that each of the functions $A = c^\pm, s^\pm, \delta, N^\pm, \alpha$ has a remarkable property:

$$\text{if } V(x) \equiv 0 \text{ at } x \geqslant b \text{ then } A(x, q) \equiv A(b, q) \text{ at } x \geqslant b \tag{9}$$

and, therefore, has a clear physical meaning: $c^\pm(b, q)$ and $s^\pm(b, q)$ are the amplitudes with which F and G are contained at $x = b$ in the wavefunctions (4*b*) non-normalized to the unit density of the flux as $x \to \infty$, and, as follows from (7)–(9), $\delta(b, q)$, $N^\pm(b, q)$ and $\alpha(b, q)$ are the phase-shift $\delta(q)$ and the normalization factors $N^\pm(q)$ and $\alpha(q)$ if $V \equiv 0$ at $x \geqslant b$.

So, for constructing the solutions $u^\pm$ of the initial problem (1) by formulae (4), one should solve problems (5) and then find the limit (8) for each function (7). However, this is not the final stage of the reformulation. A further reformulation is prompted by the following obvious facts. First, problems (5) are very simple to solve numerically in some finite interval ($x \in [0, b]$); second, in this inner region $V(x)$ is a more complicated function for use in the analytical study of equations (5*a*) than it is in the outer region ($x \in [b, \infty)$); finally, owing to (2), $V(b) \to 0$ as $b \to \infty$.

From the above-mentioned facts it seems quite reasonable to calculate $c^{\pm}$ and $s^{\pm}$ in the interval $[0, b]$ numerically, to substitute the found values $c^{\pm}(b, q)$ and $s^{\pm}(b, q)$ into equations (5*a*), and then to construct the solutions of the reformulated problems in the half-interval $[b, \infty)$ analytically, namely as limits $c^{\pm(\infty)}$ and $s^{\pm(\infty)}$ of some sequences $\{c^{\pm(m)}\}_{m=0}^{\infty}$ and $\{s^{\pm(m)}\}_{m=0}^{\infty}$ uniformly converging in the $\mathbb{C}^0$-metric [11] if $x \in [b, \infty]$, $m \to \infty$ and b is large enough. Unfortunately, all the iterations of problems (5) rewritten in an integral form generate the sets of coupled Volterra-type equations [14] which are too complicated to be analysed. To operate with uncoupled equations we realize the above-mentioned reformulation as follows.

Let $c^{\pm}$ and $s^{\pm}$ be known at some point $x = b$ and be connected with new unknown functions $y_1^{\pm}$ and $y_2^{\pm}$ by

$$\begin{Bmatrix} c^{\pm}(x, q) \\ s^{\pm}(x, q) \end{Bmatrix} = \exp\left(\begin{Bmatrix} +1 \\ -1 \end{Bmatrix} B_3(b, x, q)\right) \begin{Bmatrix} y_1^{\pm}(x, q) \\ y_2^{\pm}(x, q) \end{Bmatrix}. \tag{10a}$$

Here B_3 is one of the three integrals used below and is determined by

$$B_{l,n}(b, x, q) \equiv q^{-1} \int_b^x V(t) \left(G_l^2(\rho, \eta)\delta_{n,1} - F_l^2(\rho, \eta)\delta_{n,2} + F_l(\rho, \eta)G_l(\rho, \eta)\delta_{n,3}\right) \mathrm{d}t \tag{10b}$$

where $n = 1, 2, 3$, $\rho = tq$ and $\delta_{n,m}$ is the Kronecker symbol [12]. By substituting (10) we reduce equations (5*a*), with the boundary conditions shifted to the point $x = b$, to differential equations for $y_i^{\pm}$ given at $x = b$ by

$$y_1^{\pm}(b, q) = c^{\pm}(b, q) \qquad y_2^{\pm}(b, q) = s^{\pm}(b, q). \tag{11}$$

Then we rewrite the so obtained problems in an equivalent integral form

$$y_i^{\pm}(x, q) = y_i^{\pm(-1)}(x, q) + \langle x|\hat{P}_i(b, q)|y_j^{\pm}(t, q)\rangle. \tag{12a}$$

Hereafter $i = 1, 2$ and $j = j(i) = 1 + \delta_{i,1}$; also it is assumed that

$$y_i^{\pm(-1)}(x, q) \equiv y_i^{\pm}(b, q) \qquad x \geqslant b \tag{12b}$$

and the operators $\hat{P}_i$ map any function $z(x, q)$ into the integrals

$$\langle x|\hat{P}_i(b, q)|z(t, q)\rangle \equiv \int_b^x P_i(b, t, q)z(t, q)\,\mathrm{d}t \tag{12c}$$

$$P_i(b, x, q) \equiv \partial_x B_i(b, x, q) \exp\left((-1)^i 2B_3(b, x, q)\right). \tag{12d}$$

As early as the first iteration equations (12*a*) are diagonalized:

$$y_i^{\pm}(x, q) = y_i^{\pm(0)}(x, q) + \langle x|\hat{T}_i(b, q)|y_i^{\pm}(t, q)\rangle. \tag{13a}$$

Here, for brevity, we use the notation

$$y_i^{\pm(0)}(x,q) \equiv y_i^{\pm(-1)}(x,q) + \langle x|\hat{P}_i(b,q)|y_j^{\pm(-1)}(t,q)\rangle \tag{13b}$$

and introduce the operator products $\hat{T}_i \equiv \hat{P}_i\hat{P}_j$ in a standard way: for any function z it is assumed that $\hat{T}_i z \equiv \hat{P}_i(\hat{P}_j z)$ and

$$\langle x|\hat{T}_i(b,q)|z(t',q)\rangle \equiv \int_b^x P_i(b,x,q)\,\mathrm{d}t \int_b^t P_j(b,t',q)z(t',q)\,\mathrm{d}t'. \tag{13c}$$

According to (11), (12*b*) and (13*b*), equations (13*a*) determining y_1^+, y_2^+ or y_1^-, y_2^- are connected with each other only by the constants, which are known by assumption. These constants are the values of c^+, s^+ or c^-, s^- at the point $x = b$. Such a simple connection, achieved due to a properly chosen substitution (10), allows us to analyse all the solutions $y_i^\pm$ of equations (13) independently from each other.

2.2. *Iterations of problems (13)*

To investigate equations (13*a*) in the half-interval $[b,\infty)$ we introduce the iteration sequences $\{y_i^{\pm(m)}\}_{m=-1}^{\infty}$ defined recursively: $y_i^{\pm(-1)}$ are the constants (12*b*), $y_i^{\pm(0)}$ are the functions (13*b*), and then as the order of the index m increases ($m = 1, 2, \ldots$) we assume that $y_i^{\pm(m)}$ is the right-hand side of the relevant equation (13*a*) in which $y_i^\pm$ is replaced by $y_i^{\pm(m-1)}$. This definition together with (12*b*) and (13*c*) give us the two equivalent representations

$$y_i^{\pm(m)}(x,q) = y_i^{\pm(-1)}(b,q) + \langle x|\hat{P}_i(b,q)|y_j^{\pm(m-1)}(t,q)\rangle \tag{14a}$$

$$y_i^{\pm(m)}(x,q) = y_i^{\pm(-1)}(b,q)\sum_{p=0}^{m}\langle x|\hat{T}_i^p(b,q)|\theta(t)\rangle + y_j^{\pm(-1)}(b,q)\sum_{p=0}^{m}\langle x|\hat{T}_i^p\hat{P}_i(b,q)|\theta(t)\rangle \tag{14b}$$

where $m = 0, 1, \ldots$ and θ is the theta function [12].

2.3. *Analysis of convergence for iterations of problems (13)*

Theorem. Let $x \in [b,\infty)$, $b > 0$ and the function (2) be limited so that

$$I_l(b,x) < (1/2)\ln 3 \qquad x \in [b,\infty). \tag{15}$$

Then, the sequences $\{y_{i,l}^{\pm(m)}\}_{m=-1}^{\infty}$ uniformly converge in the $\mathbb{C}^0$-metric to the solutions $y_{i,l}^\pm$ of the problems (13) and the differences ${}^{(m)}y_{i,l}^\pm \equiv y_{i,l}^\pm - y_{i,l}^{\pm(m)}$ satisfy the inequalities

$$|{}^{(m)}y_{i,l}^\pm(x,q)| < D_{i,l,m}^\pm(b,x,q)\cosh v_l(b,x)v_l^{2m+2}(b,x)/\Gamma(2m+3) \tag{16a}$$

where $i = 1, 2$, $j = 1 + \delta_{i,1}$, $m = -1, 0, \ldots$, Γ is the gamma function [12] and

$$D_{i,l,m}^\pm(b,x,q) \equiv |y_{i,l}^{\pm(-1)}(b,q)|w_l^{i-1}(b,x,q)\left(\theta(m) + \theta(-m-1)v_l^2(b,x)/2\right) + |y_{j,l}^{\pm(-1)}(b,q)|w_l^{2-i}(b,x,q)v_l(b,x)/(2m+3) \tag{16b}$$

$$v_l(b,x) \equiv \left(\exp(2I_l(b,x)) - 1\right)/2 \tag{16c}$$

$$w_l(b,x,q) \equiv \max_{b\leqslant\rho/q\leqslant x}\left\{\left((2l+1)/(2\pi\rho^2)\right)^{1/2}G_l^2(\rho,\eta)\right\}. \tag{16d}$$

Proof. By using (2) and the Klarsfeld bounds [15]

$$F_l^2(\rho,\eta) \qquad |F_l(\rho',\eta)G_l(\rho,\eta)| < q\left(2\pi x'x/(2l+1)\right)^{1/2} \tag{17}$$

where $\rho' \equiv qx' \leqslant \rho \equiv qx$, we derive from (10*b*) the auxiliary estimates

$$|B_n(b,x,q)| < I(b,x) \qquad n = 2,3. \tag{18}$$

Maximizing $q^{-1}F^2$ and B_2, B_3 in (12*d*) with the help of (17) and (18) and then using (2), (16*c*) and the identity $\partial_x v \equiv \partial_x I \exp(2I)$ we obtain

$$|P_i(b,x,q)| < \left(\delta_{i,1}\left((2l+1)/2\pi\rho^2\right)^{1/2} G^2(\rho,\eta) + \delta_{i,2}\right)\partial_x v(b,x). \tag{19}$$

Due to (12*c*), (16*c*), (16*d*) and (19) we have

$$|\langle x|\hat{P}_i(b,q)|\theta(t)\rangle| < v(b,x)w^{2-i}(b,x,q). \tag{20}$$

To prove the relations

$$|\langle x|\hat{T}_1^p(b,q)|\theta(t)\rangle| < \int_b^x \partial_{t_1}v(b,t_1)\,\mathrm{d}t_1 \int_b^{t_1} \partial_{t_2}v(b,t_2)\,\mathrm{d}t_2 \int_b^{t_{2p-1}} \partial_{t_{2p}}v(b,t_{2p})\,\mathrm{d}t_{2p} \tag{21a}$$

$$= v^{2p}(b,x)/\Gamma(2p+1) \tag{21b}$$

in the case of $p = 1$, we put $z = \theta$ in (13*c*), maximize $q^{-1}G(qt_1,\eta)F(qt_2,\eta)$ and B_3 by the right-hand sides of (17) and (18) and then use (2), (16*c*) and the identity $\partial_x v \equiv \partial_x I \exp(2I)$. Further, using the identities $\hat{T}_1^p \equiv \hat{T}_1^{p-1}\hat{T}_1$ we prove by induction the validity of (21) for any $p = 2, 3, \ldots$.

Unfortunately, the functions $\langle x|\hat{T}_2^p(b,q)|\theta(t)\rangle$, $p = 1,2,\ldots$, cannot be estimated in an analogous way. In fact, according to (10*b*), (12*c*) and (13c) the arguments $\rho_1 = qt_1$ and $\rho_2 = qt_2$ of the Coulomb functions $F(\rho_1,\eta)$ and $G(\rho_2,\eta)$ which are comprised in the kernels of the operators $\hat{T}_2^p$, $p = 1,2,\ldots$, do not satisfy the condition $\rho_1 < \rho_2$ under which, for $|q^{-1}FG|$, inequality (17) holds. However, by virtue of the definitions $\hat{T}_i \equiv \hat{P}_i\hat{P}_j$ and $\hat{T}_i^p \equiv \hat{T}_i^{p-1}\hat{T}_i$, the equalities $\hat{T}_2^p = \hat{P}_2\hat{T}_1^{p-1}\hat{P}_1$, $p = 1,2,\ldots$ are valid. With these definitions and the estimates (18)–(21) we have

$$|\langle x|\hat{T}_2^p(b,q)|\theta(t)\rangle| < w(b,x,q)\int_b^x \partial_t v(b,t)|\langle t|\hat{T}_1^{p-1}(b,q)|\partial_{t'}v(b,t')\rangle|\,\mathrm{d}t \tag{22a}$$

$$< w(b,x,q)v^{2p}(b,x)/\Gamma(2p+1) \qquad p = 1,2,\ldots. \tag{22b}$$

To prove the bounds

$$|\langle x|\hat{T}_i^p(b,q)\hat{P}_i(b,q)|\theta(t)\rangle| < w^{2-i}(b,x,q)v^{2p+1}(b,x)/\Gamma(2p+2) \tag{23}$$

in the case $i = 1$ we first maximize the function $\langle t|\hat{P}_1(b,q)|\theta(t')\rangle$ in the identity $\hat{T}_1^p\hat{P}_1\theta \equiv \hat{T}_1^p(\hat{P}_1\theta)$ by using (20) and then apply (21*a*); in the case $i = 2$ we use (19), (21*b*) and the identity $\hat{T}_2^p\hat{P}_2 \equiv \hat{P}_2\hat{T}_1^p$.

Using (14), (16*b*), (16*c*) and the obtained results (20)–(23) we show that

$$|y_i^{\pm(n)}(x,q) - y_i^{\pm(m)}(x,q)| < D_{i,m}^{\pm}(b,x,q) \sum_{p=0}^{n-m-1} v^{2(p+m+1)}(b,x)/\Gamma\left(2(p+m+1)+1\right) \tag{24}$$

for any $i = 1, 2$, $j = 1 + \delta_{i,1}$, $n = 0, 1, \ldots$ and $m = -1, \ldots, n-1$.

Now, having all the necessary estimates, we immediately prove both the statements of the theorem. Inequalities $0 \leqslant v(b,x) < 1$ (generated by (15) and (16*c*)) and the estimates (21)–(24) imply that in the half-interval $[b,\infty)$ the operators $\hat{T}_i$ and $\hat{T}_i\hat{P}_i$ are the contracting operators [11]. Therefore, the sequences $\{y_i^{\pm(m)}\}_{m=-1}^{\infty}$ converge uniformly to the functions $y_i^{\pm}$ satisfying (12*a*) and (13*a*). Due to these facts we can put $n = \infty$ and $y_i^{\pm(\infty)} \equiv y_i^{\pm}$ in (24) in order to obtain (16), use the known power expansions [12] of the hyperbolic cosine and, thus, complete the proof.

2.4. *Iterations and estimates for problems (1) and (5)*

Let problem (5) or (13) be solved in the interval $[0,b]$ where b is such that (15) is fulfilled and the sequences $\{c^{\pm(m)}\}_{m=0}^{\infty}$ and $\{s^{\pm(m)}\}_{m=0}^{\infty}$ are defined as

$$\begin{Bmatrix} c^{\pm(m)}(x,q) \\ s^{\pm(m)}(x,q) \end{Bmatrix} \equiv \begin{Bmatrix} c^{\pm}(x,q) \\ s^{\pm}(x,q) \end{Bmatrix} = \exp\left(\begin{Bmatrix} +1 \\ -1 \end{Bmatrix} B_3(b,x,q)\right) \begin{Bmatrix} y_1^{\pm}(x,q) \\ y_2^{\pm}(x,q) \end{Bmatrix} \tag{25a}$$

if $x \leqslant b$, and

$$\begin{Bmatrix} c^{\pm(m)}(x,q) \\ s^{\pm(m)}(x,q) \end{Bmatrix} \equiv \exp\left(\begin{Bmatrix} +1 \\ -1 \end{Bmatrix} B_3(b,x,q)\theta(m-1)\right) \begin{Bmatrix} y_1^{\pm(m-1)}(x,q) \\ y_2^{\pm(m-1)}(x,q) \end{Bmatrix} \tag{25b}$$

if $b \leqslant x \leqslant \infty$. Here $m = 0, 1, \ldots$, B_3 is the integral (10*b*), $y_i^{\pm(-1)}$ are the constants (12*b*) and $y_i^{\pm(m)}$ are the functions (14). These sequences converge uniformly in the whole semi-axis $\mathbb{R}^+$ to the solutions of problems (5). The functions $c^{\pm(0)}$ and $s^{\pm(0)}$, due to (12*b*) and (25), have property (9) and are the exact solutions of these problems if the potential V is cut off at the point $x = b$. Due to the above-mentioned properties of sequences (25) two assertions are valid. First, for each function $f = \delta, N^{\pm}, \alpha, U^{\pm}, u^{\pm}$ we can find the corresponding sequence $\{f^{(m)}\}_{m=0}^{\infty}$ uniformly converging on $\mathbb{R}^+$. For this purpose, we determine the elements $f^{(m)}$ as the right-hand sides of the relevant representations (4) and (7) in which $c^{\pm}$ and $s^{\pm}$ are substituted by functions (25), i.e. we assume that at any $m = 0, 1, \ldots$ and $x \in \mathbb{R}^+$

$$\delta^{(m)}(x,q) \equiv \arctan\left(s^{+(m)}(x,q)/c^{+(m)}(x,q)\right) \tag{26a}$$

$$N^{\pm(m)}(x,q) \equiv \left(\left(c^{+(m)}(x,q)\right)^2 + \left(s^{+(m)}(x,q)\right)^2\right)^{\mp 1/2} \tag{26b}$$

$$\alpha^{(m)}(x,q) \equiv -c^{+(m)}(x,q)c^{-(m)}(x,q) - s^{+(m)}(x,q)s^{-(m)}(x,q) \tag{26c}$$

$$U^{\pm(m)}(x,q) \equiv c^{\pm(m)}(x,q)F(\rho,\eta) + s^{\pm(m)}(x,q)G(\rho,\eta) \tag{26d}$$

$$u^{\pm(m)}(x,q) \equiv N^{\pm(m)}(\infty,q)U^{\pm(m)}(x,q) + \begin{Bmatrix} 0 \\ \alpha^{(m)}(\infty,q)u^{+(m)}(x,q) \end{Bmatrix}. \tag{26e}$$

Second, the functions δ^0, $N^{\pm(0)}$, α^0, $U^{\pm(0)}$ and $u^{\pm(0)}$ thus determined have an apparent physical meaning: they are the relevant functions (4) and (7) in the case when the potential V is cut off at the point $x = b$.

Using (4), (7), (16), (25), (26) and the identities $f \equiv f^{(\infty)}$ we estimate from above the differences ${}^{(m)}f \equiv f - f^{(m)}$ for the functions $f = c^{\pm}, s^{\pm}$:

$$\left\{ \begin{array}{l} |{}^{(m)}c^{\pm}(x,q)| \\ |{}^{(m)}s^{\pm}(x,q)| \end{array} \right\} < \exp\left(\left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\} B_3(b,x,q)\theta(m-1) \right) \left\{ \begin{array}{l} D^{\pm}_{1,m-1}(b,x,q) \\ D^{\pm}_{2,m-1}(b,x,q) \end{array} \right\}$$
$$\times \cosh v(b,x) v^{2m}(b,x)/\Gamma(2m+1) \qquad m = 0, 1, \ldots \tag{27a}$$

and then for all the functions $f = \delta, N^{\pm}, \alpha, U^{\pm}, u^{\pm}$. Then we show that these estimates have the form of the asymptotic inequalities [16]

$$\begin{array}{ll} |{}^{(m)}f_l(x,q)| \leqslant \theta(x-b) \mathrm{O}\left(v^{2m+\theta(-m)}(b,x)/\Gamma(2m+1)\right) & f \neq u^{\pm} \\ |{}^{(m)}u^{\pm}_l(x,q)| < \mathrm{O}\left(v^{2m+\theta(-m)}(b,\infty)/\Gamma(2m+1)\right) & \end{array} \tag{27b}$$

if b and l are fixed and $m \to \infty$, or if m is fixed and $v_l(b,x) \to 0$ at $\forall x \geqslant b$. The validity of estimates (27b) in the first case implies that our perturbation theory is mathematically correct under condition (15). The validity of these estimates in the second case allows us to use this theory for constructing the asymptotics of functions (4) and (7) in the region $x \geqslant b$ in two limits: $l \to \infty$ at fixed q and $q \to 0$ at fixed l. To show how to do that, we first have to analyse (15) and (27).

2.5. Analysis of perturbation theory

At $x \geqslant b$ functions (2) and (16c) monotonically vanish if l is fixed and b increases:

$$I_l(b,x),\ v_l(b,x) \to 0 \qquad b \to \infty \tag{28}$$

or, if b is fixed and l increases,

$$I_l(b,x),\ v_l(b,x) = \mathrm{O}(l^{-1/2}) \qquad l \to \infty. \tag{29}$$

Therefore, inequality (15) is certainly valid in two cases. First, at any fixed l and any b exceeding the root $b_{\min}(l)$ of the equation

$$I_l\left(b_{\min}(l), \infty\right) \equiv (2\pi/(2l+1))^{1/2} \int_{b_{\min}(l)}^{\infty} t|V(t)|\,\mathrm{d}t = (\ln 3)/2 \tag{30}$$

and second, at any fixed b and any l exceeding the root

$$l_{\min}(b) \equiv \pi \left((2/\ln 3) \int_b^{\infty} t|V(t)|\,\mathrm{d}t \right)^2 - 1/2 \tag{31}$$

of the equation $I_{l_{\min}(b)}(b,\infty) = (\ln 3)/2$.

Thus, our perturbation theory can certainly be applied at $x \geqslant b > 0$ in the two above-mentioned cases. Due to (28) and (29) the estimates (27), characterizing the efficiency of this theory, are improved in the first case with increasing b and in the second case with increasing l.

The qualitative explanation of these conclusions is the following. Due to (2), in the region $x \geqslant b > 0$ the potential V is 'screened' [3] ($|V(x)| \leqslant l(l+1)x^{-2}$) by the repulsive centrifugal barrier. Hence, V is a perturbation if l is fixed and b is large ($b > b_{\min}(l)$) or, if b is fixed and l is large ($l > l_{\min}(b)$). Obviously, the degree of screening is improved in the first case with increasing b and in the second case with increasing l.

At sufficiently large l the centrifugal barrier can screen the potential, satisfying (2), everywhere, i.e. at all $x \in \mathbb{R}^+$. Therefore, it is interesting to generalize the perturbation theory to the case $b = 0$. To this end, we reconsider the proof of the theorem and all the subsequent constructions, first for the functions with the sign '+' and then for the functions with the sign '$-$'.

Let $b = 0$ and (15) be fulfilled. Then, due to (5*b*) and (11), $y_i^{+(-1)}(0, q) = \delta_{i,1}$, $i = 1, 2$. Therefore, the representations (14*b*) of $y_i^{+(m)}$ are simplified:

$$y_i^{+(m)}(x, q) = \sum_{p=0}^{m} \langle x|\hat{T}_i^p(0, q)\hat{P}_2^{i-1}(0, q)|\theta(t)\rangle \qquad m = 0, 1, \ldots \tag{32}$$

and, what is more important, contain only limited operators. Using (32) and assuming $w \equiv 1$, one can easily be convinced of the following.

For y_i^+, $y_i^{+(m)}$ and ${}^{(m)}y_i^+$ the theorem remains valid. Hence, for all of the subsequent assertions concerning the functions c^+, s^+, δ, $N^\pm$, U^+ and u^+, which have representations (4), (7) and (10) that do not contain y_1^- and y_2^-, are also valid.

Then, the relevant formulae (25) and (26) are essentially simplified. For instance, at $m = 0, 1$, they are reduced to the relations

$$c^{+(0)} = N^{\pm(0)} \equiv 1 \qquad s^{+(0)} = \delta^{(0)} \equiv 0 \qquad U^{+(0)} = u^{+(0)} \equiv F \tag{33}$$

$$c^{+(1)}(x, q) = \exp(B_3(0, x, q)) \qquad s^{+(1)}(x, q) = \langle x|\hat{P}_2(0, q)|\theta(t)\rangle / c^{+(1)}(x, q) \tag{34a}$$

$$\delta^{(1)}(x, q) = \arctan\left(\exp(-2B_3(0, x, q))\,\langle x|\hat{P}_2(0, q)|\theta(t)\rangle\right) \tag{34b}$$

$$N^{\pm(1)}(x, q) = \left(\cos\delta^{(1)}(x, q)\exp(-B_3(0, x, q))\right)^{\pm 1} \tag{34c}$$

$$U^{+(1)} = \exp(B_3(0, x, q))\left(F(\rho, \eta) + \tan\delta^{(1)}(x, q)G(\rho, \eta)\right) \tag{34d}$$

$$u^{+(1)}(x, q) = \cos\delta^{(1)}(\infty, q)\exp(B_3(x, \infty, q))\left(F(\rho, \eta) + \tan\delta^{(1)}(x, q)G(\rho, \eta)\right). \tag{34e}$$

Finally, the estimates (27*a*) for ${}^{(m)}c^+$ and ${}^{(m)}s^+$ are also simplified. For instance, at $m = 0, 1$, they have the form of

$$\begin{aligned} |{}^{(m)}c^+(x, q)| &< (1/2)\exp\left(B_3(0, x, q)\delta_{m,0}\right) v^2(0, x)\cosh v(0, x) \\ |{}^{(m)}s^+(x, q)| &< \exp\left(-B_3(0, x, q)\delta_{m,0}\right) v^{2m+1}(0, x)\cosh v(0, x)/(1 + 5\delta_{m,1}). \end{aligned} \tag{35}$$

Now if we let $b \to 0$ then (15) is valid at $b = 0$. In virtue of (5*b*) and (11) the representations (14) for $y_i^{-(m)}$ always (even at $b = 0$) contain the mappings $\hat{P}_1\theta$, $\hat{T}_2^p\theta$ and $\hat{T}_1^p\hat{P}_1\theta$ of the theta function. The estimates (20), (22) and (23) of these mappings become meaningless at $b = 0$ since $w(b, x, q) \to \infty$ if $b \to 0$. Moreover, it follows from (12*c*) and (12*d*) that $|\langle x|\hat{P}_1(b, q)|\theta(t)\rangle| \to \infty$ when $b \to 0$ and $VG^2 \not\subset \mathbb{L}^1_{[0,x_0]}$.

From the points already made, the following construction seems to be reasonable. First, by the method described in subsection 2.1 we find c^- and s^- in the interval $[0, x_0]$. Then, we apply our perturbation theory in the half-interval $[b, \infty)$. Under such a construction it is assumed in (10)–(26) that $b = x_0$, the functions $c^{-(m)}$, $s^{-(m)}$, $\alpha^{(m)}$, $U^{-(m)}$, $u^{-(m)}$ have correct asymptotics at zero and the relevant estimates (27) are valid for $\forall x \geqslant 0$.

So, if (15) is valid at $b = 0$, then the perturbation theory can be used to approximate the functions c^+, s^+, δ, $N^\pm$, U^+ and u^+ on the whole of $\mathbb{R}^+$ and to approximate the functions c^-, s^-, α, U^-, and u^- at $x \geqslant x_0$. Due to (30) and (31) the above condition is fulfilled if l is fixed and V is such that $b_{\min}(l) = 0$ or if V is an arbitrary (satisfying (2)) potential and $l > l_{\min}(0)$. In these cases formulae (32)–(35) hold.

Now, we study the non-Coulomb limit $R \to \infty$. As $V(x) \equiv R^2\bar{V}(x|R|)$ where $\bar{V}$ is the potential in the r-representation, integral (2) and, consequently, condition (15) are independent on R. This can be verified by expressing them in the variable r. The key estimates (17) are also independent on R and remain valid [15, 17] as $R \to \infty$ when $F_l(\rho, \eta) \to j_l(\rho)$ and $G_l(\rho, \eta) \to -n_l(\rho)$ [12]. Due to these facts, all the above mentioned conclusions, relations and formulae also remain valid in the non-Coulomb limit ($V_c \equiv 0$, $R = \infty$) if, beginning from (1a), one assumes $\eta = 0$, $x \equiv \rho \equiv kr$, $q \equiv 1$, $F_l(\rho, \eta) \equiv j_l(x)$, $G_l(\rho, \eta) \equiv -n_l(x)$ and one bears in mind that now $V(x) \equiv k^{-2}\bar{V}(x/k)$.

The next interesting limit is $R \to 0+$. In this case the repulsive Coulomb barrier increases ($V_c \to \infty$) and screens the potential V, first at $x \geqslant b > 0$ and then $\forall x \geqslant 0$. Unfortunately, we did not succeed in taking this effect into account within the perturbation theory. Indeed, estimates (27), characterizing its efficiency, are not improved as $R \to 0+$ because they contain only the function v which is independent of R.

Completing the analysis of the relations (15) and (27), it is useful to discuss the quality of estimates (27) and to show how they may be improved. According to the proof of the theorem, condition (15), definition (16c) of the function v and the structure of the relations (27) are generated by the key estimates (17). The latter do not contain R as a parameter and are rather rough estimates, especially as $x \to 0$ and $x \to \infty$ when [12] $F_l = \mathrm{O}\left(x^{(l+1)}qC_l(q)\right)$ and $F_l, G_l = \mathrm{O}(1)$. What is more important is that estimates (17) do not contain the Coulomb barrier factor $C_l(q) \equiv (2q)^l|\Gamma(l+1+\mathrm{i}/2q)|\exp(-\pi/4kR)$ and, therefore, they do not take into account the dependence $C_l(q) \to \infty$ as $R \to 0+$ reflecting the effect of Coulomb screening. Clearly, estimates (27) do not describe this dependence and are also rather rough. Hence, for each function (4) or (7) the approximation $f \simeq f^{(m)}$ is, indeed, more accurate than the corresponding estimate that (27) gives.

Obviously, all of these estimates can be fundamentally improved. For this purpose, instead of (17), one should use less universal but more accurate estimates that take into account the structure of F and G in a proper way. For instance, as $q \to 0$ one can use the known asymptotical ($|\eta| \to \infty$) representations (WKB-asymptotics [16], Bessel–Clifford series [18] and so on) and at $\eta = 0$ one can use the estimates [3] $j_l(x) = \mathrm{O}\left((1+x^{-1})^{-l-1}\right)$ and $n_l(x) = \mathrm{O}\left((1+x^{-1})^l\right)$.

3. Examples of application of perturbation theory

3.1. *Control and improvement of the accuracy of the calculations*

In practice, problems (5) are solved numerically not on the whole semi-axis $\mathbb{R}^+$ but on a certain finite interval $[0, b]$, i.e. the approximation $V(x) \equiv 0$ at $x \geqslant b$ is used. This is the zeroth approximation for our theory. Using this theory one can evaluate the accuracy of this approximation ${}^{(0)}f$, if necessary, one can construct any function f of (4) or (7) more exactly by formulae (25) and (26) with $m = 1$, and one can choose b so that the found function $f^{(m)}$, $m = 0, 1$ should approximate the sought one with the given absolute accuracy

ε. Such a choice is made in the usual way: the function ${}^{(m)}Q(b,x,q)$, maximizing $|{}^{(m)}f|$, is calculated by formulae (27) and compared with ε at $x \geqslant b$; if ${}^{(m)}Q(b,x,q) > \varepsilon$, then the value of b should be enlarged until the inverse equality is satisfied.

In a special case, when (15) is valid at $b = 0$, the numerical solution of equations (5) is not required and, therefore, perturbation theory becomes an effective method for analytical investigation of the initial problem (1). In this case, the construction of the functions $f^{(m)}$, $m = 0, 1, \ldots$ approximating functions (4) or (7) $\forall x \geqslant 0$ is reduced to the calculation of integrals of multiplicity not higher than $m+1$, and the accuracy of approximation $f \simeq f^{(m)}$ is controlled by the estimates (27) in which $b = 0$ is assumed. The formulae (33) and (34), determining the zeroth and first approximations of c^+, s^+, δ, $N^\pm$, U^+ and u^+, are especially simple. This allows us to derive the new estimates (36)–(42).

3.2. *Estimates for the normalization factor N^+*

Let $b = 0$ and (15) be fulfilled. Then, using (7*b*), (33)–(35) and equalities $c^+ = 1 +^{(0)} c^+$ and $s^+ =^{(0)} s^+$, we obtain for $N^+(q)$ the lower bound

$$N^+(q) \geqslant \left(1 + v^2(0,\infty)\cosh v(0,\infty)\left(1 + \cosh v(0,\infty)\left(1 + v^2(0,\infty)/4\right)\right)\right)^{-1/2}. \qquad (36)$$

Within the first approximation the more accurate lower and upper estimates

$$N^{+(1)}(q) > \exp\left(-B_3(0,\infty,q)\right)\left(1 + \exp\left(-4B_3(0,\infty,q)\right)v^2(0,\infty)\right)^{-1/2} \qquad (37a)$$

$$N^{+(1)}(q) < \exp\left(-B_3(0,\infty,q)\right) \qquad (37b)$$

are valid. To prove (37*a*) and (37*b*) we started from (26*b*) and (34*c*) respectively, and then we took into account (34*a*) and (20).

By using (7*b*), (33), (34*c*) and (39) we obtain the asymptotic estimates

$$|N_l^+(q) - 1| < \mathrm{O}(l^{-1/2}) \qquad |N_l^{+(1)}(q)/N_l^+(q) - 1| < \mathrm{O}(l^{-1}) \qquad (38)$$

determining the behaviour of $N_l^+(q)$ when q is fixed and $l \to \infty$.

Our bounds (36)–(38) allow one to estimate u_l^+ at $qx \ll 1$ when, according to (4) and (5*b*), $u_l^+ \simeq N_l^+(q)C_l(q)qx^{l+1}$. These simple estimates are very useful in an analysis of many of the approximate relations (for example, Deser *et al* [19]) containing u_l^+ as $qx \to 0$.

3.3. *Estimates for the scattering phase and amplitude as $l \to \infty$*

Let $l \to \infty$ and q be fixed. Then, as was mentioned in subsection 2.5, at any l exceeding $l_{\min}(0)$ the function $f_l = c_l^+, s_l^+, \delta_l, N_l^\pm$ can be approximated on the whole semi-axis $x \geqslant b = 0$ by the function $f_l^{(m)}$ of (33) or (34). By virtue of (29) the accuracy of an approximation like this improves with increasing l. For instance, estimates (35) take the form of the asymptotic ($l \to \infty$) inequalities

$$|{}^{(m)}c_l^+(x,q)| < \mathrm{O}(l^{-1}) \qquad |{}^{(m)}s_l^+(x,q)| < \mathrm{O}(l^{-m-1/2}) \qquad m = 0, 1. \quad (39)$$

Applying (7*a*), (33), (34*b*) and (39) we obtain the asymptotic estimates

$$\begin{aligned} &|\tan\delta_l(q)| < \mathrm{O}(l^{-1/2}) \qquad |\tan\delta_l(q) - \tan\delta_l^{(1)}(q)| < \mathrm{O}(l^{-3/2}) \\ &|\tan\delta_l^{(1)}(q)/\tan\delta_l(q) - 1| < \mathrm{O}(l^{-1}) \end{aligned} \qquad (40)$$

specifying the behaviour of the scattering phase in the limit of large l.

Using (40), the usual definition of the scattering amplitude [1,3], $A_l(q) \equiv q^{-1}\sin\delta_l(q)\ \exp(\mathrm{i}\delta_l(q))$, and our first approximation,

$$A_l^{(1)}(q) \equiv q^{-1}\sin\delta_l^{(1)}(q)\exp\left(\mathrm{i}\delta_l^{(1)}(q)\right) \tag{41}$$

we prove that at fixed q

$$|A^{(1)}(q)/A_l(q) - 1| < \mathrm{O}(l^{-1}) \qquad l \to \infty. \tag{42}$$

The estimates (40) and (42) mean that with increasing l the phase and amplitude of scattering tend to zero as the functions $\delta_l^{(1)}$ and $A_l^{(1)}$ respectively. The decrease of the scattering amplitude as $l \to \infty$ can be qualitatively explained by the effect of screening; its strict mathematical proof was first given by Klarsfeld [15]. Comparing his result

$$|A_l^B(q)/A_l(q) - 1| = \mathrm{O}(l^{-1/2}) \qquad l \to \infty$$

with our estimate (42) we see that at large l the functions $\delta_l^{(1)}$ and $A_l^{(1)}$ approximate the phase and amplitude of scattering more exactly than the functions δ_l^B and A_l^B given by the standard Born formulae [3]

$$\tan\delta_l^B(q) \equiv -q^{-1}\int_0^\infty V(t)F_l^2(\rho,\eta)\,\mathrm{d}t \equiv qA_l^B(q). \tag{43}$$

We stress that $\delta_l^{(1)} \to \delta_l^B$ and $A_l^{(1)} \to A_l^B$ as $B_{3,l}(0,x,q) \to 0$ for $\forall x \geqslant 0$. This can be verified by assuming $B_{3,l} \to 0$ in (12*c*), (12*d*), (34*b*) and (41).

3.4. Construction of low-energy representations

As is already known [1,8], the low-energy scattering of two particles is mainly determined by the behaviour of the potential tail. Therefore, at $q \ll 1$ it is necessary [20,21] to take into account the long-range potential in the whole region of large distances; this is a rather difficult task. Its solution by numerical integration of the problem (1) or even (5) is an inefficient way [21] in comparison with the construction of low-energy representations [8,22]. These representations for the functions (4) and (7) can be obtained by using the known asymptotics of the Coulomb functions [12,18] as $|\eta| \to \infty$ and our perturbation theory. We clarify this for the case $V_c > 0$. The key idea of the construction proposed below is to choose b so that at $x \leqslant b$ one could use the results of a previous paper [22], and at $x > b$ the results of the present work.

Let $q \to 0$, l be fixed and by definition

$$b \equiv x_c^p \equiv (\eta/q)^p\left(1 + (1 + l(l+1)/\eta)^{1/2}\right)^p \qquad 2/3 < p < 1 \tag{44}$$

where x_c is the Coulomb turning point [1,16]. Then, $b \ll x_c$ and the conditions $q \to 0$ and $x \ll x_c$ hold in the interval $[0, b]$ and allow one to change F, G and c^+, s^+ by the corresponding finite sums of the Bessel–Clifford series [18] and the expansions

$$\left\{\begin{matrix} c_l^\pm(x,q) \\ s_l^\pm(x,q) \end{matrix}\right\} = \left\{\begin{matrix} \left(qC_l^2(q)\right)^{(-1\pm1)/2} \\ \left(qC_l^2(q)\right)^{(+1\pm1)/2} \end{matrix}\right\} \sum_{n=0}^\infty q^{2n} \left\{\begin{matrix} c_{l,n}^\pm(x) \\ s_{l,n}^\pm(x) \end{matrix}\right\} \tag{45}$$

derived in [22], where the expansions of the functions (4*b*) and (7) as $q \to 0$ and $x \leqslant b \ll x_c$ have also been constructed. By definition (44) $b \to \infty$ as $q \to 0$, therefore, for any potential (satisfying (2)) condition (15) is fulfilled at small enough q . At this q and $x \geqslant b$ each of the functions (4) and (7) can be approximated by the function $f^{(m)}$ of (26). To construct the asymptotic ($q \to 0$) estimates of accuracy ${}^{(m)}f$ of such an approximation one should, first, consistently determine the behaviour of the functions I, v, w, $y_i^{\pm(-1)}$ and $D_{i,m}^{\pm}$ as $q \to 0$ and $x \geqslant b$, then, by formulae (27*a*), obtain asymptotic estimates for ${}^{(m)}c^+$ and ${}^{(m)}s^+$ and finally, using (4), (7) and (26), construct asymptotic estimates for all other ${}^{(m)}f$. As an example, we estimate a relative accuracy of the first approximation for the scattering phase generated by the potential

$$V(x) \simeq ax^{-d} \qquad d > 2 \qquad x \gg 1. \tag{46}$$

Let $x \geqslant b$, $q \to 0$ and l be fixed so that $2ql(l+1) \ll 1$. Due to (44), $b = \mathrm{O}(q^{-2p})$, hence, for potential (46) inequality (15) is valid at

$$q < q_{\max}(l) \simeq \left(((2l+1)/2\pi)^{1/2}\,(d-2)(\ln 3)/2|a|\right)^{1/2p(d-2)}$$

and the functions (2) and (16*c*) are such that

$$I(b,x),\, v(b,x) = \mathrm{O}(q^{2p(d-2)}). \tag{47}$$

The estimate [12] $|G(\rho,\eta)| \leqslant \mathrm{O}(q^{-1/6})$, allows us to obtain from (16*d*)

$$w(b,x,q) \leqslant \mathrm{O}(q^{2p-4/3}). \tag{48}$$

Now, with the help of (11), (12*b*) and (45), we show that

$$c^+(q,b),\, y_1^{+(-1)}(q,x) = \mathrm{O}(1) \qquad s^+(q,b),\, y_2^{+(-1)}(q,x) = \mathrm{O}\left(qC^2(q)\right). \tag{49}$$

Relations (47)–(49) give us the capability to estimate functions (16*b*) and then to obtain the asymptotic form of inequalities (27*a*):

$$\begin{aligned} |{}^{(m)}c^+(x,q)| &< \mathrm{O}\left(q^{4(d-2)(m+\theta(-m))p}/\,\Gamma(2m+2\theta(-m)+1)\right) \\ |{}^{(m)}s^+(x,q)| &< \mathrm{O}\left(q^{4(d-2)(m+1)p}/\,\Gamma(2m+2)\right) \qquad m = 0, 1, \ldots. \end{aligned} \tag{50}$$

Now, using (7*a*), (26*a*), (49) and (50), we prove the sought relation

$$|\tan\delta^{(1)}(q)/\tan\delta(q) - 1| < \mathrm{O}\left(q^{4(d-2)p}\right). \tag{51}$$

Let us, artificially, put $B_3 = 0$ in (25). Then δ^1 of (26a) coincides with δ^B of (43), and (52) results in the estimate of the relative accuracy of the Born approximation $\delta \simeq \delta^B$. This estimate is new and reproduces the more sought relation $\delta = \mathrm{O}(\delta^B)$ that was first established by Berger and Spruch [23].

4. Conclusion

The main results of this paper are the following.

We have given the *complete* construction and analysis of a perturbation theory for the one-dimensional scattering problem (1) with the rather general condition (2). For our theory we have established a sufficient condition (15), the range of applicability ($b > b_{\min}(l)$ or $l > l_{\min}(b)$ independently on whether $V_c < 0$, $V_c > 0$ or $V_c \equiv 0$) and maximum and *explicit* estimates (27) and (35) of the absolute accuracy $|^{(m)}f|$ of the approximation $f \simeq f^{(m)}$ for *each* function f investigated. We have indicated that these estimates can be improved if the structure of the Coulomb functions is taken into account in more detail. We have explained how the constructed theory can be applied in order to correctly take into account the potential V at $x \geqslant b > 0$ (in some cases, at all $x \geqslant 0$), to obtain estimates for the normalization factor N^+ and to study problem (1) in the low-energy and large angular momentum limits. By estimates (40), (42) and (51) we have demonstrated that the first approximation of our perturbation theory is more exact than the standard Born approximation (43). Throughout this work we have paid special attention to a new method of constructing the irregular solution u_l^-, an explanation of the physical meaning of each auxiliary function, an analysis of various physically limiting cases, and a derivation of useful explicit and asymptotic error estimates.

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